SOME OBSERVATIONS OF APPROXIMANTS TO FIXED POINTS OF NONEXPANSIVE NONSELF-MAPPINGS IN BANACH SPACES (Nonlinear Analysis and Convex Analysis)

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SOME OBSERVATIONS OF APPROXIMANTS TO FIXED POINTS OF NONEXPANSIVE NONSELF-MAPPINGS IN BANACH SPACES

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Abstract

Let E be a Banach space, C a nonempty closed convex subset of E, and T a nonexpansive nonself-mapping from C into E. In this paper, we study the convergence of the two sequences defined by

$$x_1 = x \in C, x_{n+1} = \alpha_n x + (1-\alpha_n)QTx_n,$$
$$y_1 = y \in C, y_{n+1} = Q(\alpha_n y + (1-\alpha_n)Ty_n), \ n = 1,2,\ldots,$$

where $0 \leq \alpha_n \leq 1$, *and* Q is a sunny nonexpansive retraction from E onto C.

1 Introduction

Let E be a Banach space, C a nonempty closed convex subset of E, and T a nonexpansive nonself-mapping from C into E such that the set $F(T)$ of all fixed points of T is nonempty. In 1998, Takahashi and Kim[8] defined two contraction mappings S_t and U_t the following: For a given $u \in C$ and each $t \in (0,1)$,

$$S_t x = tu + (1-t)QTx \quad \text{for all} \quad x \in C \tag{1.1}$$

and

$$U_t x = Q(tu + (1-t)Tx) \quad \text{for all} \quad x \in C, \tag{1.2}$$

where Q is a sunny nonexpansive retraction from E onto C. Then by the Banach contraction principle, there exists a unique element $x_t \in F(S_t)$(resp. $y_t \in F(U_t)$), i.e.

$$x_t = tu + (1-t)QTx_t \tag{1.3}$$

and

$$y_t = Q(tu + (1-t)Ty_t). \tag{1.4}$$

Also, Takahashi and Kim[8] proved that if E is a reflexive Banach space, C is a nonempty closed convex subset of E which has normal structure, T is a nonexpansive nonself-mapping from C into E satisfying the weak inwardness condition. Suppose that C is a sunny nonexpansive retract. Then $\{x_t\}$ (resp. $\{y_t\}$) defined by (1.3) (resp. (1.4)) converges strongly as $t \to 0$ to an element of $F(T)$. On the other hand, Shioji and Takahashi[7] studied the convergence of the iteration

$$x_{n+1} = \alpha_n x + (1-\alpha_n)Sx_n \text{ for } n \geq 1.$$

where x, x_1 are elements of C, S is a nonexpansive mapping from C into itself such that $F(S)$ is nonempty. They proved $\{x_n\}$ converges strongry to an element of $F(S)$.

In this paper, we deal with the strong convergence to fixed points of nonexpansive nonself-mapping T, which satisfies new boundary condition. At first, We define a new bondary condition and obtain some results with respect to new boudary condition. Further we consider two iteration schemes for T. Then we prove that the iterates converge strongly to fixed points of T.

2 Preliminaries

Throughout this paper, we denote the set of all positive integer by $\mathbf{N}$. Let E be a real Banach space with norm $\|\cdot\|$, E^* a dual space of E. The value of $x^* \in E^*$ at $x \in E$ will be denote by $\langle x, x^* \rangle$. Let C be a closed convex subset of E, and T a nonexpansive nonself-mapping from C into E. We denote the set of all fixed points of T by $F(T)$. Let D be a subset of C. A mapping Q from C into D is said to be sunny if $Q(Qx + t(x - Qx)) = Qx$ whenever $Qx + t(x - Qx) \in C$ for $x \in C$ and $t \geq 0$. A mapping Q from C into D is said to be retraction if $Q^2 = Q$. A subset D of C is said to be a sunny nonexpansive retract if there exists sunny nonexpansive retraction of C onto D. Concerning sunny nonexpansive retractions, The following lemma was proved by Bruck, Jr.[1], Reich[5]:

Lemma 2.1 *Let E be Banach space whose norm Gâteaux differentiable, C a convex subset of E, D a nonempty subset of C, and Q a retraction from C onto D. Then Q is sunny nonexpansive if and only if*

$$\langle x - Qx, J(y - Qx) \rangle \leq 0 \quad \textit{for each} \quad x \in C \quad \textit{and} \quad y \in D.$$

The modulus of convexity of E is defined by

$$\delta(\epsilon) = \inf\{1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|x - y\| \geq \epsilon\}$$

for all ϵ with $0 \leq \epsilon \leq 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for all $\epsilon > 0$. Let $U = \{x \in E : \|x\| = 1\}$. The duality mapping J from E into 2^{E^*} is defined by

$$J(x) = \{y^* \in E^* : \langle x, y^* \rangle = \|x\|^2 = \|y\|^2\}, \ x \in E.$$

The norm of E is said to be Gâteaux differentiable norm if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.5}$$

exists for each $x, y \in U$. It is also said to be uniformly Gâteaux differentiable if for each $y \in U$, the limit(2.5) is attained uniformly for $x \in U$. It is well known that if the norm of E is uniformly Gâteaux differentiable then the duality mapping is single-valued and norm weak star, uniformly continuous on each bounded subset of E. A closed convex subset C of E is said to have normal structure, if for each bounded closed convex subset K of C, which contains at least two points, there exists an element of K which is not a diametral point of K. It is well known that a closed convex subset of a uniformly convex Banach space has normal structure and a compact convex subset of Banach space has normal structure.

Let μ be a continuous, linear functional on l^∞ and let $(a_1, a_2, \ldots) \in l^\infty$. We write $\mu(a_n)$ instead of $\mu((a_1, a_2, \ldots))$. A function μ is said to be Banach limit if

$$\|\mu\| = \mu_n(1) = 1 \quad \text{and} \quad \mu_n(a_{n+1}) = \mu_n(a_n) \quad \text{for all} \quad (a_1, a_2, \ldots) \in l^\infty.$$

We know that if μ is Banach limit then

$$\liminf_{n\to\infty} a_n \leq \mu_n(a_n) \leq \limsup_{n\to\infty} a_n$$

for all $a = (a_1, a_2 \ldots) \in l^\infty$. The following lemma was proved by Shioji and Takahashi[7].

Lemma 2.2 *Let a be a real number, and $(a_1, a_2 \ldots) \in l^\infty$ such that $\mu_n(a_n) \leq a$ for all Banach limits μ and $\limsup_{n\to\infty}(a_{n+1} - a_n) \leq 0$. Then $limsup_{n\to\infty} a_n \leq a$.*

Next, we introduce several boundary conditions upon the nonself-mapping.

(i) **Rothe's condition**: $T(\partial C) \subset C$, where ∂C is boundary set of C;

(ii) **inwardness condition:** $Tx \in I_c(x)$ for all $x \in C$, where

$$I_c(x) = \{y \in E : y = x + a(z - x) \text{ for some } z \in C \text{ and } a \geq 0\};$$

(iii) **weak inwardness condition:** $Tx \in \text{cl } I_c(x)$ for all $x \in C$, where cl denotes the norm-closure; and

(iv) **nowhere normal-outward condition:** $Tx \in \{y \in E | y \neq x, Py = x\}^c$ where P is the metric projection from E onto C.

It is easily seen that there hold implications: (i)⇒(ii)⇒(iii)⇒(iv). Now, we define a new boundary condition.

Definition 2.1 (condition (C1)) *$Tx \in S_x^c$ for all $x \in C$, where Q is a sunny nonexpansive retraction from E onto C, $x \in C$, and $S_x = \{y \in E | y \neq x, Qy = x\}$.*

Remark 2.1 *Let H be a Hilbert space, C a nonempty closed convex subset of H, and T a nonexpansive nonself-mapping from C into H. Then T satisfies nowhere normal-outward condtion if and only if T satisfies condition (C1).*

By using condition (C1), we obtain two propositions.

Proposition 2.1 *Let E be a Banach space whose norm is uniformly Gâteaux differentiable, C a nonempty closed convex subset of E, T a nonexpansive nonself-mapping from C into E. Suppose that C is a sunny nonexpansive retract. and T satisfies weak inwardness condition then T satisfies condition (C1).*

Proposition 2.2 *Let E be a Banach space, C a nonempty closed convex subset of E, T a nonexpansive nonself-mapping from C into E. Suppose that C is a sunny nonexpansive retract, and T satisfies condition (C1). Then $F(T) = F(QT)$, where Q is a sunny nonexpansive retraction from E onto C.*

This proposition is very simple, but very useful. By using this proposition, we can extend all fixed point theorems with respect to nonexpansive self-mappings in Banach space, because when C is a sunny nonexpansive retract, T is a nonexpansive nonself-mapping from C into E which satisfies condition (C1), by applying fixed point theorems to QT where Q is a sunny nonexpansive retraction from E onto C, we can obtain results concerned with fixed points of QT, then we have theorems concerned with fixed points of T. On the other hand, we follow the two corollaries, the proof mainly due to Takahashi and Kim[8].

Corollary 2.1 *Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of E which has normal structure, and T a nonexpansive nonself-mapping from C into E. Suppose that C is a sunny nonexpansive retract of E, and T satisfies condition (C1), and $\{x_t\}$ the sequence defined by (1.3). Then T has a fixed point if and only if $\{x_t\}$ remains bounded as $t \to 0$ and in this case, $\{x_t\}$ converges strongly as $t \to 0$ to a fixed point of T.*

Corollary 2.2 *Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of E which has normal structure, and T a nonexpansive nonself-mapping from C into E. Suppose that C is a sunny nonexpansive retract of E, and T satisfies condition (C1), and $\{y_t\}$ the sequence defined by (1.4). Then T has a fixed point if and only if $\{y_t\}$ remains bounded as $t \to 0$ and in this case, $\{y_t\}$ converges strongly as $t \to 0$ to a fixed point of T.*

Also, by using Reich[6]'s result, and propositino 2.2, we obtain two corollaries.

Corollary 2.3 *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of E, and T a nonexpansive nonself-mapping from C into E. Suppose that C is a sunny nonexpansive retract of E, and T satisfies condition (C1), and $\{x_t\}$ the sequence defined by (1.3). Then T has a fixed point if and only if $\{x_t\}$ remains bounded as $t \to 0$ and in this case, $\{x_t\}$ converges strongly as $t \to 0$ to $Q_2u \in F(T)$ where Q_2 is the unique sunny nonexpansive retraction from C onto $F(T)$.*

Corollary 2.4 *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of E, and T a nonexpansive nonself-mapping from C into E. Suppose that C is a sunny nonexpansive retract of E, and T satisfies condition (C1), and $\{y_t\}$ the sequence defined by (1.4). Then T has a fixed point if and only if $\{y_t\}$ remains bounded as $t \to 0$ and in this case, $\{y_t\}$ converges strongly as $t \to 0$ to $Q_2u \in F(T)$ where Q_2 is the unique sunny nonexpansive retraction from C onto $F(T)$.*

3 Main Results

In this section, we study two type strong convergence of nonexpansive nonself-mappings which satisfies condition (C1). The proof mainly due to Wittmann[10], and Shioji and Takahashi[7].

Theorem 3.1 *Let E be a uniformly convex Banach space whose norm is uniforml Gâteaux differentiable, C a nonempty closed convex subset of E, and T a nonexpansive nonself-mapping from C into E such that $F(T) \neq \phi$. Suppose that C is a sunny nonexpansive retract of E, and T satisfies condition (C1). Let Q_1 be a sunny nonexpansive retraction from E onto C, $\{\alpha_n\}$ a sequence of real numbers such that $0 \leq \alpha_n \leq 1$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Suppose that $\{x_n\}$ is given by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) Q_1 T x_n \text{ for } n \geq 1.$$

Then, $\{x_n\}$ converges strongly to $Q_2x \in F(T)$, where Q_2 is a sunny nonexpansive retraction from C onto $F(T)$.

Theorem 3.2 *Let E be a uniformly convex Banach space whose norm is unifromly Gâteaux differentiable, C a nonempty closed convex subset of E, and T a nonexpansive nonself-mapping from C into E such that $F(T) \neq \phi$. Suppose that C is a sunny nonexpansive retract of E, and T satisfies condition (C1). Let Q_1 be a sunny nonexpansive retraction from E onto C, $\{\alpha_n\}$ a sequence of real numbers such that $0 \leq \alpha_n \leq 1$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Suppose that $\{y_n\}$ is given by $y_1 = y \in C$ and*

$$y_{n+1} = Q_1(\alpha_n y + (1 - \alpha_n)Ty_n) \text{ for } n \geq 1.$$

Then, $\{y_n\}$ converges strongly to $Q_2 y \in F(T)$, where Q_2 is a sunny nonexpansive retraction from C onto $F(T)$.

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